Graph States and Local Complementation in the Qutrit ZX-calculus

Xiaoyan Gong, and Quanlong Wang

Abstract— We give a modified version of the qutrit ZX-calculus, by which we represent qutrit graph states as diagrams and prove that the qutrit version of local complementation property is true if and only if the qutrit Hadamard gate $H$ has an Euler decomposition into $4\pi/3$ green and red rotations. This paves the way for studying the completeness of qutrit ZX-calculus for qutrit stabilizer quantum mechanics.

Index Terms—Graph states, qutrit, local complementation, ZX-calculus

I. INTRODUCTION

The ZX-calculus for qubits is an intuitive and powerful graphical language. It is an important branch of category quantum mechanics introduced by Coecke and Duncan [1], which allows us to explicitly formulate quantum mechanics within the overall framework of symmetric monoidal categories. This graphical language characterizes complementarity of quantum observables (typically, Pauli $Z$ and $X$ spin observables). In addition, the ZX-calculus is universal and sound for pure qubit quantum mechanics. Although the overall ZX-calculus for pure state qubit quantum mechanics is incomplete [2], it is complete for stabilizer quantum mechanics [3].

It is well-known that the theory of quantum information and quantum computation is mainly based on qubits. However, qutrits are also useful for quantum information processing. For instance, there does not exist a $(2,3)$ quantum secret threshold scheme for qubits in which each share is also a qubit, while such a scheme for qutrits does exist [4]. Moreover, better security can be achieved for quantum cryptography using higher dimensional quantum system [5]. As the same reason of introducing ZX-calculus for pure state qubit quantum mechanics, it is natural to consider a qutrit version of ZX-calculus. In fact, the ZX-calculus for qutrits was established in [10], [11] and independently introduced as a typical special case of qutrit ZX-calculus in [14].

In this paper, we treat qutrit ZX-calculus in a modified way. Comparing to the rules in [11], we add two new rules (S3) and (H2') while remove the (P2) rule which can be derived. The rules (S1) and (P1) are expanded but (K2) is reduced. In contrast to the rules in [14], the rules (S3), (P1) and (H2') are added, (S1) is expanded, but (D) is vanishing since we do not consider scalars in this paper. Also, we need not to make the assumption as adopted in [14] that any topological deformation of the internal structure does not matter. With the current version of rules in hand, one could prove some useful graphical properties of qutrits such as the Hopf law, coincidence of all dualizers and commutativity of copy and co-copy.

Our main result in this paper is to establish the equivalence of local complementation and Hadamard decomposition in the qutrit ZX-calculus. The local complementation [6], [7] is a graph transformation which is a powerful tool for the study of graph states. As a class of special stabilizer states, graph states [8] is the specific algorithm resources in one-way quantum computing model, and has broad application in quantum information processing. In [9], graph states are represented by diagrams of the qubit ZX-calculus, and it is further proved that the local complementation property (Van den Nest’s theorem) is true if and only if the Hadamard gate $H$ has an Euler decomposition into $\pi/2$ -green and red rotations. In this paper, we represent qutrit graph states by diagrams of the qutrit ZX-calculus and prove that the qutrit version of local complementation property is true if and only if the qutrit Hadamard gate $H$ has an Euler decomposition into $4\pi/3$ -green and red rotations. This paves the way for studying the completeness of qutrit ZX-calculus for qutrit stabilizer quantum mechanics.

II. QUTRIT ZX-CALCULUS

In this paper, we will work in a symmetric monoidal category and ignore non-zero scalars. First we list the rules of qutrit ZX-calculus in Figure 1, where the angles $\alpha, \beta, \eta, \theta \in [0, 2\pi)$. Note that all the diagrams should be read from top to bottom. In addition, each diagram has a standard interpretation $[\cdot]$ in the Hilbert spaces [11].

For convenience, we denote the frequently used angles $2\pi/3$ and $4\pi/3$ by 1 and 2 respectively. The box $H$ is called a Hadamard gate.

III. QUTRIT GRAPH STATES

Let $G = (V,E)$ be a graph with $n$ vertices $V$, each corresponding to a qutrit, and a collection $E$ of undirected edges connecting pairs of distinct vertices (no self loops). Multiple edges are allowed, as long as the multiplicity (weight) does not exceed 2.
**Definition 1 (Graph State):** [8] A qutrit graph state can be defined as

\[ |G\rangle = \mathcal{U}(|+\rangle^{\otimes n}), \]

Where

\[ |+\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle), \]

\[ \mathcal{U} = \prod_{l,m \in V} C_{lm}, \]

\( C_{lm} = \sum_{j=0}^{2} \sum_{k=0}^{2} \omega^{jk} |jk\rangle \langle jk|, \omega = e^{i\frac{2\pi}{3}}, \)

the \( lm \) element \( C_{lm} \) of the adjacency matrix \( \Gamma \) is the number of edges connecting vertex \( l \) with vertex \( m \).

By the standard interpretation, we can check that

\[ C_{lm} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}. \]

Graphically, we have

**Lemma 1:**

By direct calculation,

\[ C_{lm}^2 = \sum_{j,k=0}^{2} \sum_{l,m \in V} \omega^{jk} |jk\rangle \langle jk| = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}. \]

Similar to the proof of Lemma 1, we have

**Lemma 2:**

Furthermore,

\[ C_{lm}^3 = \sum_{j,k,l=0}^{2} \sum_{l,m \in V} \omega^{jk} |jk\rangle \langle jk| = I. \]

Graphically, we have

**Lemma 3:**

Now we can represent any qutrit graph state in the ZX-calculus:

**Proposition 1:** A qutrit graph state \(|G\rangle\), where \( G = (E; V) \) is an \( n \)-vertex graph, is represented in the graphical calculus as follows:

- for each vertex \( v \in V \), a green node with one output;
- for each single edge \( \{u, v\} \in E \), an \( H \) node connected to the green nodes representing vertices \( u \) and \( v \);
- for each double edge \( \{u, v\} \in E \), an \( H^2 \) node connected to the green nodes representing vertices \( u \) and \( v \).

**IV. LOCAL COMPLEMENTATION**

In this section, we prove the equivalence of local complementation and Hadamard decomposition in the qutrit ZX-calculus.

**Definition 2 (Local Complementation):** [12] Let \( G = (V, \Gamma) \) be a multiple graph with adjacency matrix \( \Gamma \) and multiplicity not exceed 2, \( \lambda \in \{1, 2\} \). The \( \lambda \)-local complementation at the vertex \( u \) is the is the multigraph \( G*_{\lambda} u = (V, \Gamma') \) such that \( \forall v, w \in V, v \neq w, \Gamma'(v, w) = [\Gamma(v, w) + \lambda \Gamma(u, v) \cdot \Gamma(u, w)](mod3). \)

For example, let \( G \) be the following graph

![Graph Diagram]
The corresponding graph state \( |G\rangle \) is

![Graph State](image)

Its 1-local complementation \( G \circlearrowleft 1 \) at the vertex 1 is

![1-local Complementation](image)

The corresponding graph state \( |G \circlearrowleft 1\rangle \) is

![Graph State](image)

**Theorem 2**: The local complementation property is true if and only if \( H \) can be decomposed as follows

This equation will be called the Euler decomposition of \( H \), as similar to the qubit case. We give some consequences of this decomposition in the following.

The rest of the paper is dedicated to the proof of Theorem 2: the equivalence of local complementation property and the Euler decomposition of \( H \). Note that below we only consider 1-local complementation, since the case of 2-local complementation is similar.

**A. Local Complementation Implies Euler Decomposition**

**Lemma 6**: Local complementation of a triangle implies the \( H \)-decomposition:

![Local Complementation](image)

**B. Euler Decomposition Implies Local Complementation**

We begin with the simplest non-trivial examples of local complementation, namely triangles (with one multiple edge). We need the following

**Lemma 7**:

![Euler Decomposition](image)

**Lemma 8**: A local complementation on the top vertex of the triangle removes the opposite edge.

![Local Complementation](image)

**Lemma 9**: A local complementation on the top vertex of another form of the triangle removes the opposite edge.

![Local Complementation](image)

**Lemma 10**: Complete graphs and Star graphs

Let \( K_n \) (\( n > 1 \)) be a complete graph with each pair of green nodes connected by \( H \) nodes. Then

![Complete Graphs and Stars](image)

The proof is quite similar to that of Lemma 8 in [9]. General case The general case can be derived from the previous case: we only need to consider the neighbors of the top vertex \( u \).
(non-neighbor part keep unchanged). According to Lemma 3, if there is no edge connecting two green vertices in the neighbors of \( u \), then it can be rewritten as connected by an \( H^\dagger \) and an \( H \)-edge; if there is an \( H \)-edge connecting two green vertices, then it can be rewritten as connected by two \( H^\dagger \)-edges. Therefore, the neighbor part of \( u \) can be rewritten as a complete graph with each pair of green nodes connected by \( H^\dagger \) nodes and some more \( H^\dagger \) or \( H \) edges that can be pushed to the bottom of the graph. We can do similar things to the edges connecting \( u \) and its neighbors. Thus a general graph can be rewritten as a Star graph with more \( H^\dagger \) or \( H \) edges that can be applied after local complementation of the star graph.

For \( \lambda = 2 \), we can prove in a similar way that Euler decomposition and local complementation are equivalent.

V. CONCLUSION AND FURTHER WORK

In this paper, we modify the rules of qutrit ZX-calculus by adding (S3) and (H2'), expanding (S1) and (P1), removing (P2) and reducing (K2) in comparing with that of [11]. With these rules, we prove some useful graphical properties of qutrits such as the Hopf law, coincidence of all dualizers and commutativity of copy and co-copy. As a main result, we represent qutrit graph states by diagrams of the qutrit ZX-calculus and prove that the qutrit version of local complementation property is true if and only if the qutrit Hadamard gate \( H \) has an Euler decomposition into \( 4\pi/3 \)-green and red rotations.

Given this modified version of the qutrit ZX-calculus, it is natural to consider the completeness of qutrit ZX-calculus for qutrit stabilizer quantum mechanics. It would also be interesting to apply the qutrit ZX-calculus to contextuality problem since quantum contextuality exists only in dimensions greater than two [16].

REFERENCES